Quantum statistical mechanics
(4 Problems, 60 possible points)

Due on: April 9, 5:00 pm

(Email me TeX'd solutions, or scan your homework and email it to me. If you have problems finding a way to turn in your homework in these unusual times, please let me know as soon as possible!)

Note: An important part of science is communicating your understanding to other people. That is to say, a solution that may be technically correct but which I (the grader) cannot understand is not much better than an incorrect solution. So, please answer the following questions neatly, clearly, and logically – Thanks!

Problem 1 (10 points): Practice with density matrix formalism

We ended section 5.4 of the notes with a few examples in which we calculated the density matrix for simple examples. In this problem we'll do another:

Consider the hamiltonian for a simple two-dimensional quantum mechanical rotor,

$$\mathcal{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta},$$

where $0 \leq \theta \leq 2\pi$, and eigenstates must satisfy $\psi(\theta) = \psi(\theta + 2\pi)$.

(A): What are the eigenstates of this Hamiltonian? What are the energy levels?

(B): What are the elements of the density matrix, $\langle\theta'|\rho|\theta\rangle$, in the canonical ensemble? Give simplified expressions for these elements in the high- and low-temperature limits.

Problem 2 (10 points): Canonical ensemble density matrix for non-interacting, distinguishable particles

In Section 5.5 we said that the product states are appropriate for *distinguishable* particles, and in Section 5.6 we looked at the canonical ensemble density matrix for non-interacting particles where the eigenstates were either the fermionic or bosonic (anti)-symmetrizations of the product states. Study the density matrix and canonical partition function using the product states. **Show that if you do so, you do not get Gibbs' correction factor of $\frac{1}{N!}$.**

Additionally, **show that there are no spatial correlations between the particles in the system, and so there is no statistical interparticle potential.**

Problem 3 (20 points): Bose condensation in other dimensions

Consider an ideal gas of non-interacting, spinless bosons in a box whose generalized volume is $V \equiv L^d$ in d dimensions. We'll be working in the grand canonical ensemble.

(A): Given the value of the chemical potential μ, Calculate both the grand potential $\mathcal{G} = -k_B T \mathcal{Q}$, the number density $n = N/V$. Write your answer in terms of d and the $f_m^+(z)$ functions. *Hints: integration by parts will help with the expression for* $\log \mathcal{Q}$. *The volume of a d-dimensional unit sphere is* $V_n = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$

(B): Calculate the ratio PV/E, and compare with the classical result for an ideal gas.

(C): Calculate the critical temperature, $T_c(n)$, for Bose-Einstein condensation in d dimentions.

(D): Calculate the heat capacity at low temperatures, $C(T)$ for $T < T_c(n)$.

(E): Calculate the heat capacity at high temperature. What is the ratio $C_{max}/C(T \to \infty)$?

(F): How does the ratio you calculated in the last part behave as $d \to 2$? What does this say about the dimensions for which your calculation was valid?

Problem 4 (20 points): Heat capacities of an ideal Fermi gas

(A): Show that for an ideal Fermi gas one has

$$\frac{1}{z}\left.\frac{\partial z}{\partial T}\right|_P = -\frac{5}{2T}\frac{f_{5/2}^-(z)}{f_{3/2}^-(z)}$$

.

(B): From the lecture notes we know the grand partition function for this system. Take the appropriate thermodynamic derivatives to derive the entropy of an ideal Fermi gas.

(C): Find an expression for $\gamma \equiv C_P/C_V$, the ratio of the heat capacities at constant pressure and at constant volume. Your answer should be given in terms of combinations of $f_m^-(z)$.

Check that the low temperature limit of your expression is

$$\gamma \approx 1 + \frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2$$

Question: (0 points): Measurement of homework difficulty

How much time did you spend on this homework? Feel free to answer either in absolute terms (i.e., number of hours worked) or in qualitative terms relative to the average homework from last semester. Thanks!

Question: (0 points): Lecture survey

Please fill out the (tiny, extremely short, and painless) survey from the course webpage about the lectures. Much appreciated!