Probability problem set
(3 Problems, 60 possible points)

Due on: Feb. 28, 5:00 pm

(Return to my mailbox in N212, hand to me personally, or email me TeX'd solutions)

> Note: An important part of science is communicating your understanding to other people. That is to say, a solution that may be technically correct but which I (the grader) cannot understand is not much better than an incorrect solution. So, please answer the following questions neatly, clearly, and logically – Thanks!

Problem 1 (15 points): Detailed balance

Let's consider functions which satisfy the Boltzmann equation, and for simplicity let's assume that the external potential $U(\boldsymbol{q}) = 0$. As we saw in class, one way to make the collision integral vanish is by ensuring that f_1 satisfies the *detailed balance* condition:

$$f_1^{eq}(\boldsymbol{r}, \boldsymbol{p}_1')f_1^{eq}(\boldsymbol{r}, \boldsymbol{p}_2') = f_1^{eq}(\boldsymbol{r}, \boldsymbol{p}_1)f_1^{eq}(\boldsymbol{r}, \boldsymbol{p}_2),$$

where the challenge is to figure out how to guarantee this equality for all momenta.

> **The Boltzmann distribution:**
>
> By taking the log of the detailed balance condition, argue based on your knowledge of conserved quantities during collisions that f_1^{eq} takes the form of the Maxwell-Boltzmann distribution. (i.e., interestingly, having the collision term in the Liouville equation makes the f_1 sit in the Boltzmann distribution at equilibrium.).

Problem 2 (15 points): Momentum moments

Let's consider a gas of N particles of mass m, in thermal equilibrium at temperature T in a box of volume V.

(a) Write down the appropriate equilibrium one-particle density, $f_{eq}(\boldsymbol{q}, \boldsymbol{p})$

(b) What is the joint characteristic function, $\langle \exp(-i\boldsymbol{k} \cdot \boldsymbol{p}) \rangle$**?**

(c) Calculate the joint cumulants, $\left\langle p_x^l p_y^m p_z^n \right\rangle_c$**, for any integers** l, m, n**.**

(d) Calculate the joint moment, $\langle p_\alpha p_\beta \boldsymbol{p} \cdot \boldsymbol{p} \rangle$**, for any choice of cartesian directions** α, β **(hint: the answer is given in the next question... just show me how to get there).**

Problem 3 (30 points): Local equilibrium

If we ignore the "streaming term" $\{H_1, f_1\}$ then there is a much larger class of distributions that satisfy detailed balance, and they are said to be in *local equilibrium*; they take the form of a distribution with a Boltzmann distribution, but where the number density, temperature, and velocity all become functions of the spatial coordinates[1].

Let's use the Boltzmann formalism to calculate the thermal conductivity of a dilute gas. Suppose we have such a gas between two plates, which are a distance h apart. The first plate (at $z = 0$) is fixed at temperature T_1, and the second plate (at $z = h$) is fixed at temperature T_2. There is no net drift velocity of the gas (i.e., there are no net flows), and so our zeroth-order approximation for the one-particle density is

$$f_1^0(\boldsymbol{p}, x, y, z) = \frac{n(z)}{(2\pi m k_B T(z))^{3/2}} \exp\left[-\frac{\boldsymbol{p} \cdot \boldsymbol{p}}{2 m k_B T(z)}\right],$$

where the superscript 0 indicates that this is the zeroth-order approximation.

Local pressure

If the gas initially has no drift velocity, what is needed for a relationship between $n(z)$ and $T(z)$ to ensure that the gas velocity remains zero? You will need this relationship in later parts.

First order approximation

The zeroth-order approximation we started with does not allow for the relaxation of density and temperature variations. Let's linearize the Boltzmann equation in the single-collision-time approximation as

$$\mathcal{L}[f] \approx \left[\frac{\partial}{\partial t} + \frac{p_z}{m}\frac{\partial}{\partial z}\right] f_1^0 \approx \frac{f_1^1 - f_1^0}{\tau_K},$$

[1] Take, for instance, a glass of water with a ice cube melting in it. This is clearly not an equilibrium state, but if you look at the distribution of velocities of water molecules as a function of distance from the ice cube, you will find to a good approximation that at every point in space and at any moment in time the distribution is close to Maxwell-Boltzmann

where τ_K is again of the order of the typical time between collisions. What is your improved approximation for f_1^1, the first-order approximate solution to the Boltzmann equation (assuming it is independent of time)?

Heat transfer

What is the z-component of the heat transfer vector, h_z, defined as

$$h_z = n \left\langle c_z \frac{mc^2}{2} \right\rangle ,$$

where $\boldsymbol{c} = \boldsymbol{p}/m - \boldsymbol{u}$? Give the answer using both the zeroth order approximation, f_1^0, and using the first order approximation you derived, f_1^1. It may help you to know that

$$\left\langle p^2 \right\rangle^0 = 3mk_BT,\ \left\langle p^4 \right\rangle^0 = 15(mk_BT)^2,\ \left\langle p_z^2 p^4 \right\rangle^0 = 35(mk_BT)^3,$$

where $\langle A \rangle^0$ means the local average using f_1^0. Your answer should include a spatial derivative of the temperature.

Thermal conductivity and temperature profile

From the connection between thermal conductivity and temperature gradient, $\boldsymbol{h} = -K\nabla T$, and your answer above, what is your first-order approximation for the thermal conductivity of a dilute gas? What is the temperature profile in the steady state, $T(z)$?

Question: (0 points): Measurement of homework difficulty

How much time did you spend on this homework? Feel free to answer either in absolute terms (i.e., number of hours worked) or in qualitative terms relative to the average homework from last semester. Thanks!

Question: (0 points): Lecture survey

Please fill out the (tiny, extremely short, and painless) survey from the course webpage about the lectures. Much appreciated!